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APPLICATION OF THE PSEUDOSPECTRAL METHOD TO THERMOHYDRODYNAMIC LUBRICATION

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SUMMARY

The pseudospectral method is used for the first time to solve the thermohydrodynamic lubrication equations for a slider bearing. The orthogonal polynomials used in the series expansions are Lagrangian interpolants derived from a Legendre basis. Exponential convergence to exact solutions is demonstrated and favourable comparisons with previous work are made.

KEY WORDS: pseudospectral method; thermohydrodynamic lubrication; spectral method

1. INTRODUCTION

The thermohydrodynamic (THD) lubrication equations have been solved by numerous investigators using mainly the finite difference method (see e.g. Reference 1) and to a lesser extent the finite element method (see e.g. Reference 2; see also Khonsari's review for a more thorough discussion of numerical techniques for the THD equations[3]). Both these methods, as traditionally applied, are characterized by algebraic convergence to the exact solution as the number of grid points is increased. Pseudospectral methods, on the other hand, are typically characterized by exponential convergence as the number of grid (or collocation) points is increased. Another advantage of the pseudospectral method is that the solution is in the form of a functional relationship between calculated values and independent variables—as opposed to values only at discrete grid points—making postprocessing particularly convenient. Until now, no one has applied the pseudospectral technique to the solution of thermohydrodynamic lubrication problems. The formulation presented here will serve as a foundation for future efforts to apply the pseudospectral method to lubrication problems where the advantages of spectral methods are particularly beneficial; namely, in the solution of thermal elastohydrodynamic lubrication problems, where high resolution is required.

Description, analysis and references to applications of the pseudospectral method are discussed at length in the monograph by Boyd.[4] Examples of fluid dynamic problems that have been solved with this method range from Taylor–Couette flow,[5] driven cavity flow,[6,7] to flow past an aerofoil,[8] along with a host of other published solutions not referenced here. Although a pseudospectral code is generally more difficult to implement initially, once the necessary subroutines for calculating basis

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functions and their derivatives at grid points have been written, the algorithm is relatively easily modified for other problems. For a given accuracy, pseudospectral codes are potentially much more efficient than other codes because of the smaller linear systems to be solved.

Excellent discussions and summaries of previous research in THD lubrication have been delivered elsewhere and therefore will not be repeated here. Khonsari,[3] Szeri[9] and Pinkus[10] in particular provide background to THD analysis.

In this paper the pseudospectral method is formulated and applied to the steady thermohydrodynamic lubrication equations for a plane slider bearing. Section 2 describes the geometry and governing equations, Section 3 details the formulation of the numerical method, Section 4 presents results and comparisons with previous solutions and Section 5 concludes the work.

2. GOVERNING EQUATIONS

The geometry and co-ordinate system for a slider bearing analysed in this paper are shown in Figure 1. The bearing half-width is L; since the pressure distribution is symmetric about the bearing midplane, the co-ordinate origin is placed at the midplane so that $0 \leqslant z \leqslant L$. Note that the dimensions are not shown to scale; the length and width of bearings are typically of the order of 1000 times larger than the distance between surfaces.

Momentum, continuity and pressure equations

The basic assumptions for lubrication flows are as follows.

1a. Pressure is invariant across the fluid film (in the y-direction).
2a. Inertia forces are negligible.
3a. Velocity gradient in all but the cross-stream (y-) direction are negligible.

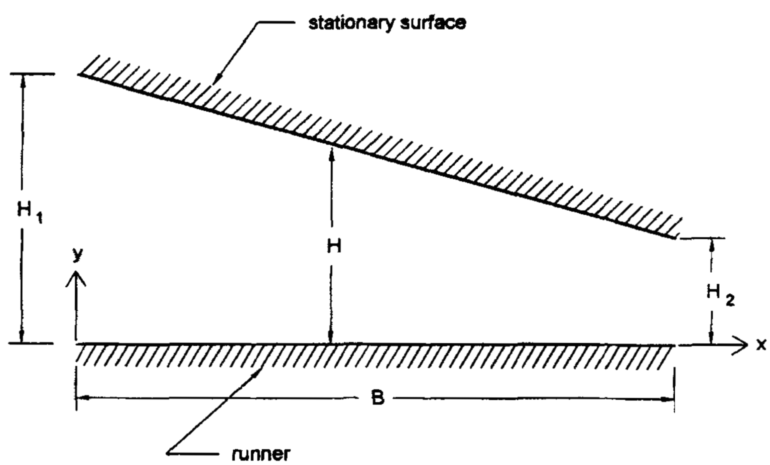

Figure 1. Geometry and co-ordinate system for slider bearing (not to scale). The z-direction is perpendicular to the plane of the page, with the co-ordiante origin at the bearing midplane and $z = L$ at the bearing edge

In addition, the flow is assumed to be steady. The Navier–Stokes equations thus reduce to

$$\frac{\partial p}{\partial x} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right), \tag{1}$$

$$\frac{\partial p}{\partial z} = \frac{\partial}{\partial y}\left(\mu \frac{\partial w}{\partial y}\right). \tag{2}$$

The continuity equation is

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \tag{3}$$

and the boundary conditions are

$$u = U \quad \text{and} \quad v = w = 0 \quad \text{at } y = 0, \qquad u = v = w = 0 \quad \text{at } y = H. \tag{4}$$

Combining equations (1)–(3) and applying the boundary conditions leads to the steady generalized Reynolds equation for pressure (see Reference 11 for mathematical details)

$$\frac{\partial}{\partial x}\left[(F_2 + G_1)\frac{\partial p}{\partial x}\right] + \frac{\partial}{\partial z}\left[(F_2 + G_1)\frac{\partial p}{\partial z}\right] = \frac{\partial}{\partial x}\left(U\frac{F_3 + G_2}{F_0} - UG_3\right), \tag{5}$$

where

$$F_0 = \int_0^H \frac{\mathrm{d}y}{\mu}, \qquad F_2 = \int_0^H \frac{\rho y}{\mu}\left(y - \frac{F_1}{F_0}\right)\mathrm{d}y, \qquad F_1 = \int_0^H \frac{y}{\mu}\,\mathrm{d}y, \qquad F_3 = \int_0^H \frac{\rho y}{\mu}\,\mathrm{d}y, \tag{6}$$

$$G_1 = \int_0^H \left[y\frac{\partial \rho}{\partial y}\left(\int_0^y \frac{\zeta}{\mu}\,\mathrm{d}\zeta - \frac{F_1}{F_0}\int_0^y \frac{\mathrm{d}\zeta}{\mu}\right)\right]\mathrm{d}y, \qquad G_2 = \int_0^H \left(y\frac{\partial \rho}{\partial y}\int_0^y \frac{\mathrm{d}\zeta}{\mu}\right)\mathrm{d}y,$$
$$G_3 = \int_0^H y\frac{\partial \rho}{\partial y}\,\mathrm{d}y. \tag{7}$$

Energy equation

The assumptions for the energy equation are as follows.

1b. Conduction terms other than across the fluid film (in the y-direction) are negligible.
2b. Thermal conductivity and specific heat are constant.

These assumptions, combined with assumptions 1a and 3a from above, lead to the following form of the energy equation.[10]

$$\rho c_\mathrm{p}\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}\right) = k\frac{\partial^2 T}{\partial y^2} + \beta T\left(u\frac{\partial p}{\partial x} + w\frac{\partial p}{\partial z}\right) + \mu\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]. \tag{8}$$

Ezzat and Rohde[1] showed that the calculated temperature is nearly constant in the z-direction for a finite slider bearing, so, to save computational effort, a form of equation (8) is used which is averaged over the z-direction:[12]

$$c_\mathrm{p}\left((\rho u)_\mathrm{avg}\frac{\partial T}{\partial x} + (\rho v)_\mathrm{avg}\frac{\partial T}{\partial y}\right) = k\frac{\partial^2 T}{\partial y^2} + \beta T\gamma_\mathrm{avg} + \phi_\mathrm{avg}, \tag{9}$$

where

$$(\rho u)_{\text{avg}} = \frac{1}{L}\int_0^L \rho u \, \mathrm{d}z, \qquad (\rho v)_{\text{avg}} = \frac{1}{L}\int_0^L \rho v \, \mathrm{d}z, \tag{10}$$

$$\lambda_{\text{avg}} = \frac{1}{L}\int_0^L \left(u\frac{\partial p}{\partial x} + w\frac{\partial p}{\partial z}\right) \mathrm{d}z, \tag{11}$$

$$\phi_{\text{avg}} = \frac{1}{L}\int_0^L \mu\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] \mathrm{d}z. \tag{12}$$

The boundary conditions are

$$T(0, y) = T(x, 0) = T_{\mathrm{R}}, \qquad T(x, H) = T_{\mathrm{S}} \quad \text{or} \quad \frac{\partial T}{\partial y}(x, H) = 0. \tag{13}$$

The last boundary condition in equation (13) is that for an adiabatic upper boundary.

Viscosity and density are related to temperature and pressure via the relationships[13,14]

$$\mu = \mu_1 \exp[-\alpha(T - T_{\mathrm{R}}) + \gamma p], \tag{14}$$

$$\rho = \rho_1 \exp[-\beta(T - T_{\mathrm{R}})], \tag{15}$$

where μ_1 and ρ_1 are the inlet viscosity and density respectively, α and γ are viscosity coefficients and β is the lubricant thermal expansivity.

Next the governing equations are non-dimensionalized to facilitate comparison with other published results and to establish a computational domain suitable for use with our choice of orthogonal basis functions. We non-dimensionalize the governing equations using the scales

$$\begin{gathered}
\frac{x}{B} = \frac{s+1}{2}, \qquad \frac{y}{H} = \frac{r+1}{2}, \qquad \frac{z}{B} = \frac{t+1}{2}\frac{L}{B}, \qquad \bar{H} = \frac{H}{H_2}, \\
\bar{u} = \frac{u}{U}, \qquad \bar{v} = \frac{v}{UH_2/B}, \qquad \bar{w} = \frac{w}{U}, \\
\bar{p} = \frac{p}{\mu_1 UB/H_2^2}, \qquad \bar{T} = \frac{T}{T_{\mathrm{R}}}, \qquad \bar{\mu} = \frac{\mu}{\mu_1}, \qquad \bar{\rho} = \frac{\rho}{\rho_1}.
\end{gathered} \tag{16}$$

The new independent variables s, r and t are chosen in order to transform the computational domain to $[-1, 1]$ in each direction. This transformation allows the use of a Legendre polynomial basis in the pseudospectral formulation as described in the following section. Noting that r is a function of both x (since H is a function of x) and y, equation (5) becomes

$$I_1\left(\frac{\partial^2 \bar{p}}{\partial s^2} + \frac{B^2}{L^2}\frac{\partial^2 \bar{p}}{\partial t^2}\right) + \frac{\partial I_1}{\partial s}\frac{\partial \bar{p}}{\partial s} + \frac{B^2}{L^2}\frac{\partial I_1}{\partial t}\frac{\partial \bar{p}}{\partial t} = \frac{1}{2}\frac{\partial I_2}{\partial s}, \tag{17}$$

where

$$I_1 = \bar{h}^3(\bar{F}_2 + \bar{G}_1), \qquad I_2 = \bar{h}\left(\frac{\bar{F}_3 + \bar{G}_2}{\bar{F}_0} - \bar{G}_3\right), \tag{18}$$

$$\bar{F}_0 = \frac{1}{2}\int_{-1}^{1} \frac{dr}{\bar{\mu}}, \qquad \bar{F}_2 = \frac{1}{4}\int_{-1}^{1} \frac{\bar{\rho}(r+1)}{\bar{\mu}}\left(\frac{r+1}{2} - \frac{\bar{F}_1}{\bar{F}_0}\right) dr, \qquad \bar{F}_1 = \frac{1}{4}\int_{-1}^{1} \frac{r+1}{\mu}\, dr,$$

$$\bar{F}_3 = \frac{1}{4}\int_{-1}^{1} \frac{\bar{\rho}(r+1)}{\bar{\mu}}\, dr, \tag{19}$$

$$\bar{G}_1 = \frac{1}{2}\int_{-1}^{1}\left[(r+1)\frac{\partial\bar{\rho}}{\partial r}\left(\bar{f}_1 - \frac{\bar{F}_1}{\bar{F}_0}\bar{f}_0\right)\right] dr, \quad \bar{G}_2 = \frac{1}{2}\int_{-1}^{1}(r+1)\frac{\partial\bar{\rho}}{\partial r}\bar{f}_0\, dr, \quad \bar{G}_3 = \frac{1}{2}\int_{-1}^{1}(r+1)\frac{\partial\bar{\rho}}{\partial r}\, dr, \tag{20}$$

$$\bar{f}_0 = \frac{1}{2}\int_{-1}^{r} \frac{d\zeta}{\bar{\mu}}, \qquad \bar{f}_1 = \frac{1}{4}\int_{-1}^{r} (\zeta+1)\frac{d\zeta}{\bar{\mu}}. \tag{21}$$

The boundary conditions for the non-dimensional pressure are

$$\bar{p}(-1,t) = \bar{p}(1,t) = \bar{p}(s,1) = \frac{\partial\bar{p}}{\partial t}(s,-1) = 0. \tag{22}$$

The equations for $\bar{u}$ and $\bar{w}$ become

$$\bar{\mu}\frac{\partial^2\bar{u}}{\partial r^2} + \frac{\partial\bar{\mu}}{\partial r}\frac{\partial\bar{u}}{\partial r} = \frac{\bar{H}^2}{2}\frac{\partial\bar{p}}{\partial s}, \tag{23}$$

$$\bar{\mu}\frac{\partial^2\bar{w}}{\partial r^2} + \frac{\partial\bar{\mu}}{\partial r}\frac{\partial\bar{w}}{\partial r} = \frac{\bar{H}^2}{2}\frac{B}{L}\frac{\partial\bar{p}}{\partial t}. \tag{24}$$

Once the pressure distribution has been calculated by solving equation (17), velocities must be calculated for use in the energy equation. Equations (23) and (24) are solved for $\bar{u}$ and $\bar{w}$. The equation for $\bar{v}$ could be obtained by solving the continuity equation for $\bar{v}$, but numerically this leads to the physically unreasonable condition of non-zero $\bar{v}$ on the stationary member surface owing to the imposition of a boundary condition solely on the lower surface. The problem is alleviated by differentiating the continuity equation with respect to y and then solving the resulting second-order equation with homogeneous boundary conditions on both surfaces, similar to the procedure followed by Boncompaign *et al.*[15] The validity of the procedure has been confirmed by comparing computed solutions for $\bar{v}$ with the exact solution for an infinitely wide isothermal slider bearing. For example, the RMS error for $N = M = 14$ on an 1111 uniform grid was $3{\cdot}210^{-6}$ solving the continuity equation and $1{\cdot}610^{-6}$ solving the differentiated form of the continuity equation. The resulting equation for the quantity $\bar{\rho}\bar{v}$ is thus

$$\frac{\partial^2\bar{\rho}\bar{v}}{\partial r^2} = -\bar{H}\left(\frac{\partial^2\bar{\rho}\bar{u}}{\partial s\partial r} + \frac{B}{L}\frac{\partial^2\bar{\rho}\bar{w}}{\partial r\partial t}\right) + \frac{d\bar{H}}{ds}\left((r+1)\frac{\partial^2\bar{\rho}\bar{u}}{\partial r^2} + \frac{\partial\bar{\rho}\bar{u}}{\partial r}\right). \tag{25}$$

The velocity boundary conditions are

$$\bar{u} = 1 \quad \text{and} \quad \bar{v} = \bar{w} = 0 \quad \text{at } r = -1, \qquad \bar{u} = \bar{v} = \bar{w} = 0 \quad \text{at } r = 1. \tag{26}$$

The energy equation becomes

$$Pe\left((\bar{\rho}\bar{u})_{avg}\frac{\partial\bar{T}}{\partial s}-\frac{r+1}{\bar{H}}\frac{d\bar{H}}{ds}(\bar{\rho}\bar{u})_{avg}\frac{\partial\bar{T}}{\partial r}+\frac{(\bar{\rho}\bar{v})_{avg}}{\bar{H}}\frac{\partial\bar{T}}{\partial r}\right)-\frac{2}{\bar{H}^2}\frac{\partial^2\bar{T}}{\partial r^2}-\frac{PrEc}{2}\bar{\beta}\bar{T}\bar{\lambda}_{avg}=\frac{PrEc}{2}\frac{1}{\bar{H}^2}\bar{\phi}_{avg}, \quad (27)$$

where

$$(\bar{\rho}\bar{u})_{avg}=\frac{1}{2}\int_{-1}^{1}\bar{\rho}\bar{u}\,dt, \qquad (\bar{\rho}\bar{v})_{avg}=\frac{1}{2}\int_{-1}^{1}\bar{\rho}\bar{v}\,dt, \quad (28)$$

$$\bar{\lambda}_{avg}=\int_{-1}^{1}\left(\bar{u}\frac{\partial\bar{p}}{\partial s}+\frac{B}{L}\bar{w}\frac{\partial\bar{p}}{\partial t}\right)dt, \quad (29)$$

$$\bar{\phi}_{avg}=2\int_{-1}^{1}\bar{\mu}\left[\left(\frac{\partial\bar{u}}{\partial r}\right)^2+\left(\frac{\partial\bar{w}}{\partial r}\right)^2\right]dt. \quad (30)$$

The Peclet number is

$$Pe=\frac{\rho_1 c_p U H_2^2}{kB} \quad (31)$$

and the product of the Prandtl and Ecert numbers is

$$PrEc=\frac{\mu_1 U^2}{kT_R}. \quad (32)$$

The boundary conditions for temperature become

$$\bar{T}(-1,r)=\bar{T}(s,-1)=1, \qquad \bar{T}(s,1)=\frac{T_S}{T_R} \quad \text{or} \quad \frac{\partial\bar{T}}{\partial r}(s,1)=0. \quad (33)$$

The equations for viscosity and density become

$$\bar{\mu}=\exp[-\bar{\alpha}(\bar{T}-1)+\bar{\gamma}\bar{p}], \quad (34)$$

$$\bar{\rho}=\exp[-\bar{\beta}(\bar{T}-1)], \quad (35)$$

where

$$\bar{\alpha}=\alpha T_R, \qquad \bar{\gamma}=\gamma\frac{\mu_1 UB}{H_2^2}, \qquad \bar{\beta}=\beta T_R. \quad (36)$$

Three performance parameters are now defined for future reference. The first is a non-dimensional load parameter

$$\bar{W}=\frac{W}{2L\mu_1 U}\left(\frac{H_2}{B}\right)^2, \quad (37)$$

which can be expressed as

$$\bar{W}=\frac{1}{4}\int_{-1}^{1}\int_{-1}^{1}\bar{p}\,ds\,dt, \quad (38)$$

and the second and third are non-dimensional inlet and outlet flow rates

$$\bar{Q}_{\text{in}} = \frac{Q_{\text{in}}}{2L\rho_1 UH_1}, \qquad \bar{Q}_{\text{out}} = \frac{Q_{\text{out}}}{2L\rho_1 UH_1}, \tag{39}$$

which can be expressed as

$$\bar{Q}_{\text{in}} = \frac{1}{4}\int_{-1}^{1}\int_{-1}^{1} \bar{\rho}(-1,r,t)\bar{u}(-1,r,t)\,\mathrm{d}r\,\mathrm{d}t, \qquad \bar{Q}_{\text{out}} = \frac{1}{4\bar{H}_1}\int_{-1}^{1}\int_{-1}^{1} \bar{\rho}(1,r,t)\bar{u}(1,r,t)\,\mathrm{d}r\,\mathrm{d}t. \tag{40}$$

The final equations to be solved are equation (17) for pressure, equations (23)–(25) for velocities and equations (27) for temperature.

3. FORMULATION AND SOLUTION PROCEDURE

The pseudospectral method proceeds by expanding the unknowns in series of orthogonal polynomials, substituting the expansions into the governing equations and boundary conditions, satisfying the governing equations at a set of collocation points and enforcing the boundary conditions at appropriate boundary points. The resulting system of linear equations is then solved for the series coefficients. For this work a series of Lagrangian interpolants constructed from Legendre polynomials is chosen. A Legendre basis is use because the collocation points are the same as those used in the Gauss–Lobatto quadrature for evaluating the integrals in the pressure equation. The Lagrangian interpolant formulation results in direct calculation of the grid point values for the unknowns, which avoids the necessity of performing Legendre transforms in the solution process.

Velocities, pressure, temperature, viscosity and density are expanded in series of Lagrangian interpolants which satisfy $h_i(s_j) = \delta_{ij}$;[16]

$$h_j(s) = -\frac{(1-s^2)L'_M(s)}{M(M+1)L_M(s_j)(s-s_j)}. \tag{41}$$

The quantity $L_M(s)$ is the Mth-order Legendre polynomial and s_j is the jth root of

$$(1-s^2)L'_M(s). \tag{42}$$

The roots of the Legendre polynomial derivatives re not known in closed form and must be calculated numerically. Expressions for $h_j(r)$ and $h_j(t)$ are similar to that for $h_j(s)$, with r or t replacing s and N or L replacing M. The unknowns are thus expanded as

$$\begin{gathered}
\bar{u} = \sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{l=0}^{L}\bar{u}_{mnl}h_m(s)h_n(r)h_l(t), \qquad \bar{v} = \sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{l=0}^{L}\bar{v}_{mnl}h_m(s)h_n(r)h_l(t), \\
\bar{w} = \sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{l=0}^{L}\bar{w}_{mnl}h_m(s)h_m(r)h_l(t), \\
\bar{\mu}\sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{l=0}^{L}\bar{\mu}_{mnl}h_m(s)h_n(r)h_l(t), \qquad \bar{\rho} = \sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{l=0}^{L}\bar{\rho}_{mnl}h_m(s)h_n(r)h_l(t), \\
\bar{p} = \sum_{m=0}^{M}\sum_{l=0}^{L}\bar{p}_{ml}h_m(s)h_l(t), \qquad \bar{T} = \sum_{m=0}^{M}\sum_{n=0}^{N}\bar{T}_{mn}h_m(s)h_n(r),
\end{gathered} \tag{43}$$

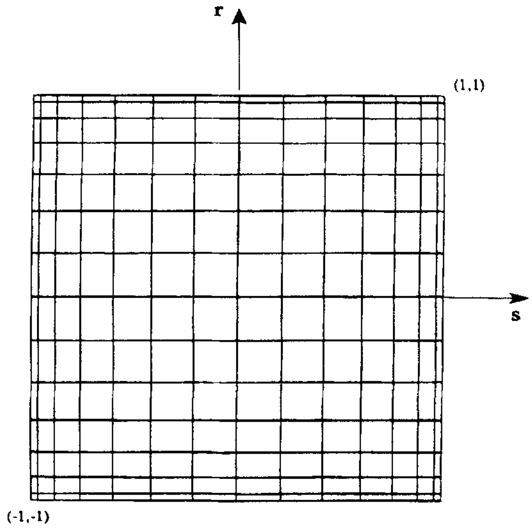

Figure 2. Grid in computational domain. The t-direction is perpendicular to the plane of the page and $-1 \leqslant t \leqslant 1$. Here $M = N = 14$

where, for example, $\bar{u}_{mnl}$ is the x-direction velocity at collocation point s_m, r_n, t_l. The collocation points are the roots of equation (42). A sample grid in the computational domain is shown in Figure 2.

Substitution of the expansion for pressure from equation (43) into equation (17) leads to the following equation at collocation point s_i, t_k:

$$\sum_{m=0}^{M} \sum_{l=0}^{L} A_{ikml} \bar{p}_{ml} = \frac{1}{2} \frac{\partial i_2}{\partial s} \Big|_{ik}, \tag{44}$$

where

$$A_{ikml} = \delta_{kl} \left(I_{1_{ik}} h''_m(s_i) + \frac{\partial I_1}{\partial s}\bigg|_{ik} h'_m(s_i) \right) + \delta_{im} \frac{B^2}{L^2} \left(I_{1_{ik}} h''_l(t_k) + \frac{\partial I_1}{\partial t}\bigg|_{ik} h'_l(t_k) \right). \tag{45}$$

The first derivatives $h'_j(s_i)$ of the Lagrangian interpolants are given by[16]

$$\frac{\mathrm{d}h_j}{\mathrm{d}s}(s_i) = \begin{cases} \dfrac{L_m(s_i)}{L_M(s_j)(s_i - s_j)}, & i \neq j, \\ 0, & i = j \neq 0, M, \\ -\dfrac{M(M+1)}{4}, & i = j = 0, \\ \dfrac{M(M+1)}{4}, & i = j = M, \end{cases} \tag{46}$$

and the second derivatives $h_j''(s_i)$ are given by[17]

$$\frac{d^2h_j}{ds^2}(s_i)=\begin{cases}\dfrac{-2L_M(s_i)}{L_M(s_j)(s_i-s_j)^2}, & i\neq j, i\neq 0, M,\\[2mm] \dfrac{-1^{M+1}M(M+1)(s_i-s_j)-4(-1)^M}{2L_M(s_j)(s_i-s_j)^2}, & i\neq j, i=0,\\[2mm] \dfrac{M(M+1)}{2L_M(s_j)(s_j)(s_i-s_j)}-\dfrac{2}{L_M(s_j)(s_j)(s_i-s_j)^2}, & i\neq j, i=M,\\[2mm] \dfrac{L_M''(s_i)}{3L_M(s_j)}, & i=j.\end{cases} \tag{47}$$

The derivatives for I_1 and I_2 are obtained by expanding the functions in terms of Lagrangian interpolants and differentiating, resulting in

$$\left.\frac{\partial I_1}{\partial s}\right|_{ik}=\sum_{m=0}^{M} I_{1_{mk}}h_m'(s_i), \tag{48}$$

$$\left.\frac{\partial I_1}{\partial t}\right|_{ik}=\sum_{l=0}^{L} I_{1_{il}}h_l'(t_k), \tag{49}$$

$$\left.\frac{\partial I_2}{\partial s}\right|_{ik}=\sum_{m=0}^{M} I_{2_{mk}}h_m'(s_i). \tag{50}$$

To evaluate the integrals in equations (19)–(21), we employ Gauss–Lobatto quadrature. The quadrature points are the roots of equation (42) and the weights are[18]

$$\omega_i=\frac{2}{M(M+1)[L_M(s_i)]^2},\quad i=0,\ldots,M. \tag{51}$$

The integrals in equation (19) thus become

$$\begin{aligned}\bar F_0(s_i,t_k)&=\frac{1}{2}\sum_{n=0}^{N}\frac{\omega_n}{\bar\mu_{ink}}, & \bar F_2(s_i,t_k)&=\frac{1}{4}\sum_{n=0}^{N}\frac{\bar\rho_{ink}(r_n+1)}{\bar\mu_{ink}}\left(\frac{r_n+1}{2}-\frac{\bar F_1(s_i,t_k)}{\bar F_0(s_i,t_k)}\right)\omega_n,\\ \bar F_1(s_i,t_k)&=\frac{1}{4}\sum_{n=0}^{N}\frac{\omega_n(r_n+1)}{\bar\mu_{ink}}, & \bar F_3(s_i,t_k)&=\frac{1}{4}\sum_{n=0}^{N}\frac{\omega_n(r_n+1)\bar\rho_{ink}}{\bar\mu_{ink}}.\end{aligned} \tag{52}$$

The integrals in equation (20) become

$$\bar G_1(s_i,t_k)=\frac{1}{2}\sum_{n=0}^{N}\left.\frac{\partial\bar\rho}{\partial r}\right|_{ink}(r_n+1)\left(\bar f_1-\frac{\bar F_1(s_i,t_k)}{\bar F_0(s_i,t_k)}\bar f_0\right)\omega_n,\ \bar G_2(s_i,t_k)=\frac{1}{2}\sum_{n=0}^{N}\omega_n(r_n+1)\bar f_0\left.\frac{\partial\bar\rho}{\partial r}\right|_{ink},$$

$$\bar G_3(s_i,t_k)=\frac{1}{2}\sum_{n=0}^{N}\omega_n(r_n+1)\left.\frac{\partial\bar\rho}{\partial r}\right|_{ink}, \tag{53}$$

where

$$\left.\frac{\partial \bar{\rho}}{\partial r}\right|_{ink} = \sum_{n=0}^{N} \bar{\rho}_{ink} h'_n(r_j). \tag{54}$$

The values for $\bar{f}_0$ and $\bar{f}_1$ in equation (21) are determined by solving the equations

$$\frac{\partial \bar{f}_0}{\partial r} = \frac{1}{2\bar{\mu}}, \qquad \frac{\partial \bar{f}_1}{\partial r} = \frac{r+1}{4\bar{\mu}} \tag{55}$$

subject to the boundary conditions $\bar{f}_0 = \bar{f}_1 = 0$ at $r = -1$. Expanding the functions in terms of Lagrangian interpolants gives

$$\bar{f}_0 = \sum_{n=0}^{N} \bar{f}_{0_n} h_n(r), \qquad \bar{f}_1 = \sum_{n=0}^{N} \bar{f}_{1_n} h_n(r) \tag{56}$$

and substituting into equation (55) leads to

$$\sum_{n=0}^{N} \bar{f}_{0_n} h'_n(r_j) = \frac{1}{2\bar{\mu}_{ijk}}, \qquad \sum_{n=0}^{N} \bar{f}_{1_n} h'_n(r_j) = \frac{r_j+1}{4\bar{\mu}_{ijk}}. \tag{57}$$

Applying the boundary conditions and then writing equation (57) for all j, where $2 \leqslant j \leqslant N$, leads to a linear system of equations for the unknowns at each s_i, t_k.

Writing equation (4) for each interior grid point and applying the boundary conditions leads to the linear system of equations to be solved for the pressure coefficients.

Equations (23)–(25) are one-dimensional boundary value problems for the velocities. Substitution of the appropriate expansions from equation (43) into equation (23) yields, for grid point s_i, r_j, t_k,

$$\sum_{n=0}^{N} B_{ijkn} \bar{u}_{ink} = \frac{\bar{H}^2(s_i)}{2} \left.\frac{\partial \bar{p}}{\partial s}\right|_{ik}, \tag{58}$$

where

$$B_{ijkn} = \bar{\mu}_{ijk} h''_n(r_j) + \left.\frac{\partial \bar{\mu}}{\partial r}\right|_{ijk} h'_n(r_j), \tag{59}$$

$$\left.\frac{\partial \bar{\mu}}{\partial r}\right|_{ink} = \sum_{n=0}^{N} \bar{\mu}_{ink} h'_n(r_j), \qquad \left.\frac{\partial \bar{p}}{\partial s}\right|_{ik} = \sum_{m=0}^{M} \bar{p}_{mk} h'_m(s_i). \tag{60}$$

Applying the boundary conditions at $r = -1$ and 1 and writing equation (58) for each grid point r_j, $1 \leqslant j \leqslant N-1$, leads to a system of $N-1$ equations for the $N-1$ unknown velocities at a particular grid point s_i, t_k. Substitution of the appropriate expansions from equation (43) into equation (24) yields, for grid point s_i, r_j, t_k,

$$\sum_{n=0}^{N} B_{ijkn} \bar{w}_{ink} = \frac{\bar{H}^2(s_i)}{2} \frac{B}{L} \left.\frac{\partial \bar{p}}{\partial t}\right|_{ik}. \tag{61}$$

The procedure for solving for the values of $\bar{w}$ is identical to that used for $\bar{u}$. Substitution of the appropriate expansions from equation (43) into equation (25) yields, for grid point s_i, r_j, t_k,

$$\sum_{n=0}^{N} h''_n(r_j)(\bar{\rho}\bar{v})_{ink} = D_{ijk}, \tag{62}$$

where

$$D_{ijk} = -\bar{H}(s_i)\left(\frac{\partial^2\bar{\rho}\bar{u}}{\partial r\partial s}\bigg|_{ijk} + \frac{B}{L}\frac{\partial^2\bar{\rho}\bar{w}}{\partial r\partial t}\bigg|_{ijk}\right) + \frac{\mathrm{d}\bar{H}(s_i)}{\mathrm{d}s}\left((r_j+1)\frac{\partial^2\bar{\rho}\bar{u}}{\partial r^2}\bigg|_{ijk} + \frac{\partial\bar{\rho}\bar{u}}{\partial r}\bigg|_{ijk}\right). \tag{63}$$

The derivatives in equation (63) are discretized as

$$\begin{aligned}\frac{\partial^2\bar{\rho}\bar{w}}{\partial r\partial t}\bigg|_{ijk} &= \sum_{n=0}^{N}\sum_{l=0}^{L}\bar{\rho}_{inl}\bar{w}_{inl}h_n'(r_j)h_l'(t_k), & \frac{\partial^2\bar{\rho}\bar{u}}{\partial r\partial t}\bigg|_{ijk} &= \sum_{m=0}^{M}\sum_{n=0}^{N}\bar{\rho}_{mnk}\bar{u}_{mnk}h_m'(s_i)h_n'(r_j),\\ \frac{\partial^2\bar{\rho}\bar{u}}{\partial r^2}\bigg|_{ijk} &= \sum_{n=0}^{N}\bar{\rho}_{ink}\bar{u}_{ink}h_n''(r_j), & \frac{\partial\bar{\rho}\bar{u}}{\partial r}\bigg|_{ijk} &= \sum_{n=0}^{N}\bar{\rho}_{ink}\bar{u}_{ink}h_n'(r_j).\end{aligned} \tag{64}$$

The energy equation becomes, at grid point s_i, r_j,

$$E_{ijmn}\bar{T}_{mn} = \frac{PrEc}{2}\frac{1}{\bar{H}^2(s_i)}\bar{\phi}_{\mathrm{avg}_{ij}}, \tag{65}$$

where

$$\begin{aligned}E_{ijmn} =& Pe\left((\bar{\rho}\bar{u})_{\mathrm{avg}_{ij}}h_m'(s_i)\delta_{jn} - \frac{r_j+1}{\bar{H}(s_i)}\frac{\mathrm{d}\bar{H}(s_i)}{\mathrm{d}s}(\bar{\rho}\bar{u})_{\mathrm{avg}_{ij}}\delta_{im}h_n'(r_j) + \frac{(\bar{\rho}\bar{v})_{\mathrm{avg}_{ij}}}{\bar{H}(s_i)}\delta_{im}h_n'(r_j)\right)\\ &- \frac{2}{\bar{H}^2(s_i)}\delta_{im}h_n''(r_j) - \frac{PrEc}{2}\bar{\beta}\bar{\lambda}_{\mathrm{avg}_{ij}}\delta_{im}\delta_{jn},\end{aligned} \tag{66}$$

$$(\bar{\rho}\bar{u})_{\mathrm{avg}_{ij}} = \frac{1}{2}\sum_{k=0}^{L}\omega_k\bar{\rho}_{ijk}\bar{u}_{ijk}, \qquad (\bar{\rho}\bar{v})_{\mathrm{avg}_{ij}} = \frac{1}{2}\sum_{k=0}^{L}\omega_k\bar{\rho}_{ijk}\bar{v}_{ijk}, \tag{67}$$

$$\bar{\lambda}_{\mathrm{avg}_{ij}} = \sum_{k=0}^{L}\omega_k\left(\bar{u}_{ijk}\sum_{m=o}^{M}\bar{p}_{mk}h_m'(s_i) + \frac{B}{L}\bar{w}_{ijk}\sum_{l=0}^{L}\bar{p}_{il}h_l'(t_k)\right), \tag{68}$$

$$\bar{\phi}_{\mathrm{avg}_{ij}} = 2\sum_{k=0}^{L}\omega_k\bar{\mu}_{ijk}\left[\left(\sum_{n=0}^{N}\bar{u}_{ink}h_n'(r_j)\right)^2 + \left(\sum_{n=0}^{N}\bar{w}_{ink}h_n'(r_j)\right)^2\right]. \tag{69}$$

The solution procedure begins by assuming a temperature and pressure distribution in order to obtain the viscosity and density from equations (34) and (35). These values are then used in the solution of the pressure equation (44). Solution of equations (58), (61) and (62) for the velocities occurs next. Finally the energy equation (65) is solved for a new temperature distribution. The process is then repeated until convergence is obtained. The sum of the normalized RMS differences between the values of pressure, all three velocity components and temperature at the current and previous iterations is used as a stopping criterion, with a value of 1×10^{-3} used for all calculation results presented here. The number of iterations is dependent upon the problem parameters and boundary conditions but is typically between 10 and 15. Solution of the linear systems is performed using Gaussian elimination. For a discretization where the truncation is the same in all directions (i.e. $N = M = L$), the operation count is $O(N^6)$ per iteration. With $N = M = L = 14$ the computation time is 250 s per iteration on an IBM RS 6000 workstation.

4. RESULTS

Although the truncations M, N and L need not have the same values, they are set equal to each other for all calculations discussed hereafter.

The formulation was first tested by comparing with several exact solutions for the infinitely wide slider (see Appendix for exact solutions from Reference 19). Figure 3 shows the convergence of the numerical to the exact solutions for $\bar{p}, \bar{u}$ and $\bar{v}$ for a slider operating under isothermal conditions. Exponential convergence is evident in the plot. The energy equation formulation was also tested by comparing with exact solutions to the energy equation for flow between parallel plates for both constant temperature and adiabatic boundaries (see Appendix). Again agreement was excellent.

Comparisons with results of Hahn and Kettleborough,[13] who solved the thermohydrodynamic problem for an infinitely wide slider, are shown in Table I. Note that Hahn and Kettleborough incorrectly incorporated the compressibility term in the energy equation (the last term on the left side of the equals sign in equation (27)) as they discussed in an appendix to their paper, so their results for case 5 are slightly in error. Comparisons were also made with results of Zienkiewicz for parallel sliders[20] and inclined sliders[21] and of Rodkiewicz *et al.*[22] and agreement was excellent.

Comparison is made next with analytical solutions to the isothermal finite slider problem. In Table II the present numerical results and results of Szeri and Powers[23] for various values of $\bar{H}_1$ and B/L are presented. Agreement is within three significant figures.

As mentioned by Boyd,[4] another check for convergence is to evaluate the magnitude of the spectral coefficients associated with the highest-frequency components in the solution. The solution error will be roughly the same order of magnitude as these highest-order coefficients. For the present formulation the solution can be expanded using a Legendre basis, employing the Legendre transform as described by Canuto *et al.*[18] and Schumack.[17] Figures 4 and 5 show how the Legendre coefficients for pressure and temperature, $\hat{p}_{ml}$, and $\hat{T}_{mn}$, decrease with increasing indices for the case described in the caption.

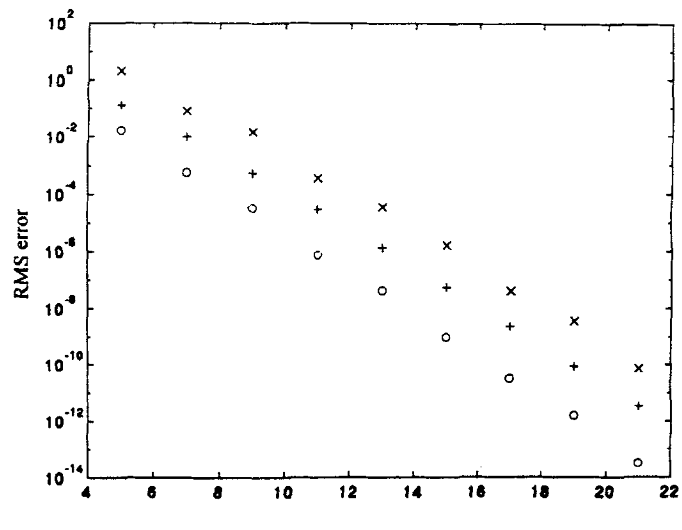

Figure 3. Convergence of pressure and velocity solutions to exact solutions for infinitely wide isothermal slider with $\bar{H}_1 = 2$: ○, $\bar{p}$; +, $\bar{u}$; ×, $\bar{v}$

Table I. Comparison of non-dimensional pressure solutions of Hahn and Kettleborough[13] with present solutions for infinitely wide slider. The cases correspond to those in Reference 13: case 2, $\bar{\alpha} = 0, \bar{\beta} = 0.036, \bar{\gamma} = 0, \bar{\lambda} = 0$; case 3, $\bar{\alpha} = 2.5, \bar{\beta} = 0.036, \bar{\gamma} = 0, \bar{\lambda} = 0$; case 4, $\bar{\alpha} = 2.5, \bar{\beta} = 0.036, \bar{\gamma} = 2.0164, \bar{\lambda} = 0$; case 5, $\bar{\alpha} = 2.5, \bar{\beta} = 0.036, \bar{\gamma} = 2.0164, \bar{\lambda} \neq 0$. Truncation for case 2 was $M = N = L = 13$; truncation for remaining cases was $M = N = L = 15$

	$x = 0.2$		$x = 0.4$		$x = 0.6$		$x = 0.8$	
Case	Reference 13	Present	Reference 13	Present	Reference 13	Present	Reference 13	Present
2	0.102	0.10196	0.193	0.19336	0.252	0.25235	0.229	0.22893
3	0.071	0.07066	0.115	0.11536	0.132	0.13263	0.107	0.10747
4	0.072	0.07306	0.122	0.12338	0.141	0.14312	0.111	0.11348
5	0.072	0.07309	0.117	0.12322	0.133	0.14282	0.106	0.11328

Table II. Comparison of non-dimensional performance parameters of Szen and Powers[23] with present solutions for finite isothermal slider. Truncation used in present solution is $M = N = L = 17$

	$\bar{W}$		$\bar{Q}_{in}$		$\bar{Q}_{out}$	
	Reference 23	Present	Reference 23	Present	Reference 23	Present
$\bar{H}_1 = 1.2, B/L = 8$	0.00401	0.004005	0.4940	0.494244	0.4224	0.422185
$\bar{H}_1 = 5, B/L = 8$	0.00897	0.008950	0.4698	0.469554	0.1198	0.119294
$\bar{H}_1 = 1.2, B/L = 0.5$	0.0636	0.063635	0.4609	0.460862	0.4494	0.449549
$\bar{H}_1 = 5, B/L = 0.5$	0.0895	0.089533	0.2161	0.216105	0.1602	0.160786

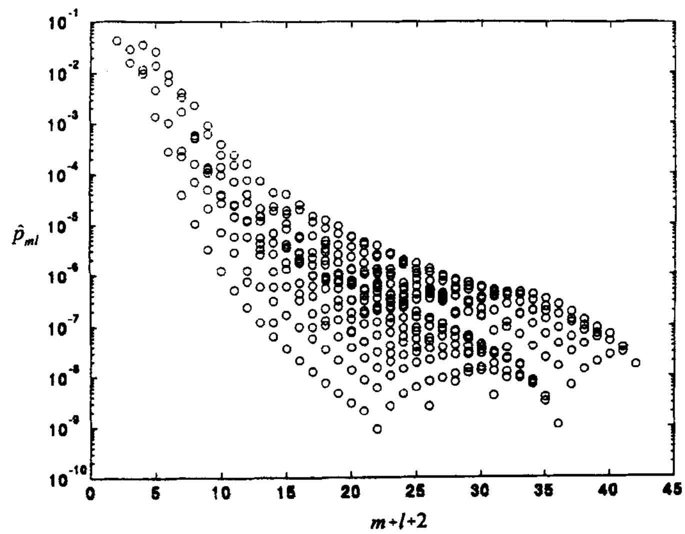

Figure 4. Legendre coefficients for pressure solution for case where $\bar{H}_1 = B/L = 2, Pe = 19.1, PrEc = 18.1$, $T_R = T_S = 100\ ^\circ F, \bar{\alpha} = 2.5, \bar{\gamma} = 0, \bar{\beta} = 0, M = N = L = 20$

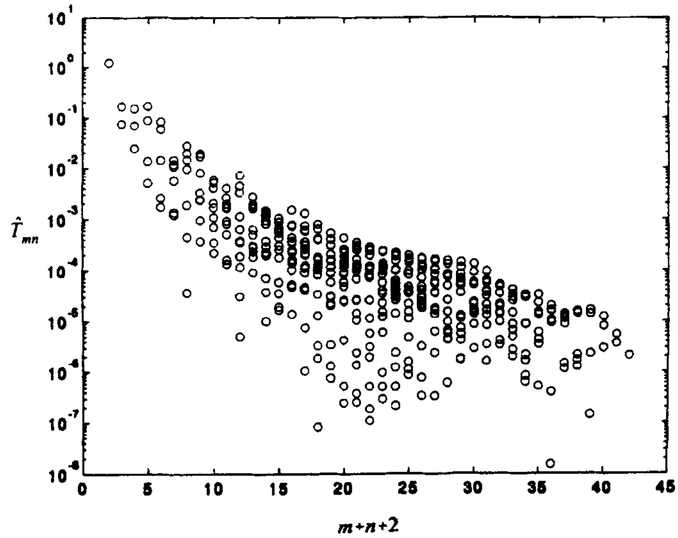

Figure 5. Legendre coefficients for temperature solustion for same case as Figure 4

Figure 6 shows results for the non-dimensional pressure at the bearing midplane for three cases: isothermal, constant boundary temperatures and constant slider temperature with adiabatic top. The results for constant boundary temperatures agree well with those of Ezzat and Rohde.[1] The figure shows that the inclusion of thermal effects has a profound effect on pressure. Pressures are lowest for the case of an adiabatic upper boundary, where the temperatures inside the fluid film are highest.

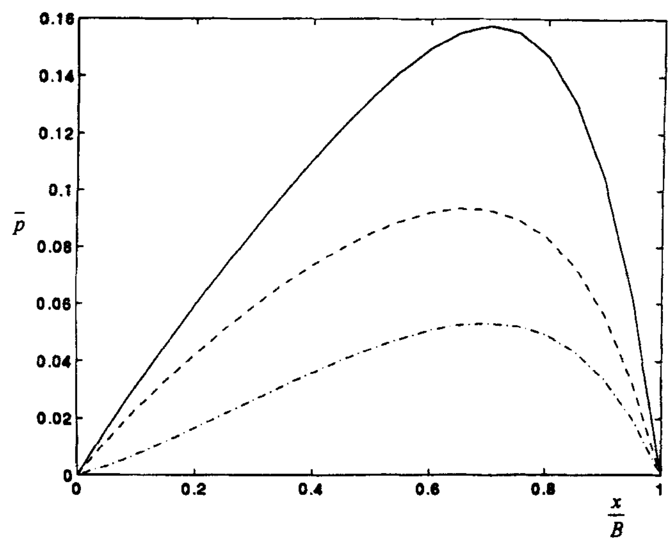

Figure 6. Pressure along bearing midplane ($z = 0$) for case where $\bar{H}_1 = B/L = 2, Pe = 19{\cdot}1$, $PrEc = 18{\cdot}1$, $T_R = T_S = 100$ °F, $\bar{\alpha} = 2{\cdot}5$, $\bar{\gamma} = 0$, $\bar{\beta} = 0{\cdot}036$, $M = N = L = 14$. Results for three sets of boundary conditions are shown: ———, isothermal slider, — — —, $T_R = T_S = 100$ °F; – · –, $T_R = 100$ °F, adiabatic stationary surface. Density variation is neglected in order to compare with results from Reference 1. The non-zero value for $\bar{\beta}$, however, still affects the pressure gradient term in the energy equation

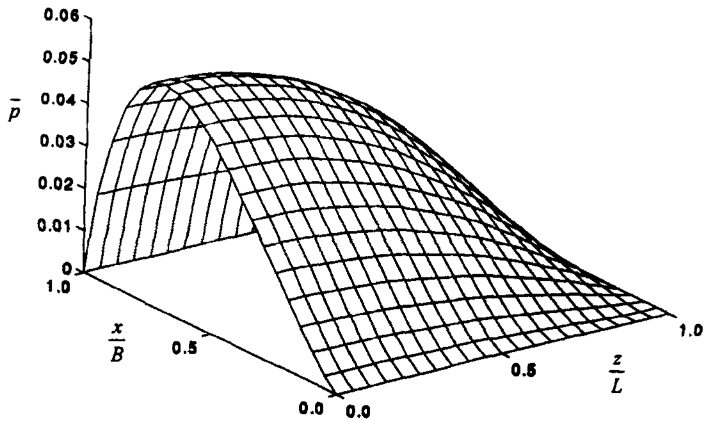

Figure 7. Two-dimensional pressure distribution same case as Figure 6. Results are for $T_R = 100$ °F, adiabatic stationary surface. Density variation is neglected as discussed in caption to Figure 6

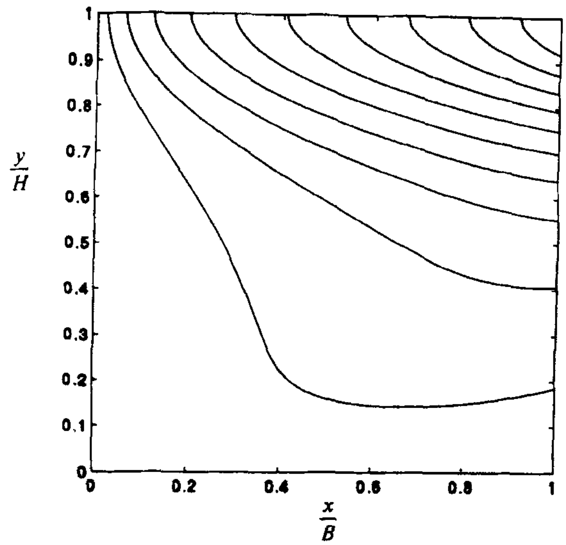

Figure 8. Temperature contours for same case as Figure 6. Results are for $T_R = 100$ °F, adiabatic stationary surface. Values for contour lines increase in increments of 0·1 from 1·1 in lower left to 2·0 in upper right of figure. Density variation is neglected as discussed in caption to Figure 6

Figure 7 shows the two-dimensional pressure distribution for the case with an adiabatic upper boundary and Figure 8 shows temperature contours for the same case. The temperature increases from the inlet to a maximum at the exit near the upper boundary, similar to the results obtained by Hahn and Kettleborough[13] for the case where heat transfer into the stationary member was modelled.

5. CONCLUSIONS

The thermohydrodynamic lubrication equations have been solved using the pseudospectral method. The solutions exhibit the exponential convergence characteristics of spectral methods. Benefits of the pseudospectral method over other computational methods include high accuracy with relatively few grid points and a solution which is a function of the independent variables and not just grid point values (making postprocessing relatively simple). The formulation presented here serves as a foundation for future efforts to apply the pseudospectral method to thermal elastohydrodynamic lubrication problems, where high resolution is required.

APPENDIX

The analytical solutions for pressure and velocity for the infinitely wide slider bearing are given by Pinkus and Sternlicht[19] and are repeated here. The solution for the non-dimensional pressure in terms of the non-dimensional x-co-ordinate $\bar{x} = x/B$ is

$$\bar{p} = \frac{6\bar{x}(1-\bar{x})}{(\bar{H}_1^2 - 1)[\bar{H}_1/(\bar{H}_1 - 1) - \bar{x}]^2}.$$

The equations for the non-dimensional velocities in terms of the non-dimensional y-co-ordinate $\bar{y} = y/H$ are

$$\bar{u} = \frac{1}{2}\frac{\mathrm{d}\bar{p}}{\mathrm{d}\bar{x}}\bar{H}^2\bar{y}(\bar{y}-1) + (1-\bar{y}), \qquad \bar{v} = \frac{\bar{H}_2}{B}\frac{\mathrm{d}\bar{H}}{\mathrm{d}\bar{x}}\left((\bar{y}^2 - \bar{y}^3) + \frac{\bar{H}^2\bar{y}^2}{2}\frac{\mathrm{d}\bar{p}}{\mathrm{d}\bar{x}}(\bar{y}-1)\right).$$

For flow between infinite parallel plates the energy equation reduces to

$$k\frac{\mathrm{d}^2T}{\mathrm{d}y^2} = -\mu\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2.$$

The solution to this equation for constant surface temperatures is

$$\bar{T} = \frac{\mu U^2}{2kT_\mathrm{R}}(\bar{y} - \bar{y}^2) + \left(\frac{T_\mathrm{S}}{T_\mathrm{R}} - 1\right)\bar{y} + 1.$$

For a constant runner temperature and adiabatic stationary surface the temperature distribution is

$$\bar{T} = \frac{\mu U^2}{2kT_\mathrm{R}}\bar{y}(2-\bar{y}) + 1.$$

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